Unit 6: Differentiation

6.1 Basic differentiation rules

(i) $\frac{\mathrm{d}}{\mathrm{d}x}k = 0$, where k is a constant

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$, where n is real (Power Rule)

(iii) $\frac{\mathrm{d}}{\mathrm{d}x}[kf(x)] = k\frac{\mathrm{d}}{\mathrm{d}x}f(x)$, where k is a constant

(iv) $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}g(x)$ (Sum and Difference Rules)

(v) $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)$ (Product Rule)

(vi) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{g^2(x)}$, where $g(x) \neq 0$ (Quotient Rule)

Proofs of the above differentiation rules are not required.

6.2 Differentiation of composite functions and inverse functions

Teachers should use examples to help students to grasp the basic concepts of composite functions, inverse functions, explicit functions and implicit functions.

Composite function can be simply viewed as a function of the values of another function.

<u>Example 1</u> .
The number of boxes n of mushrooms stored in a warehouse will be rotten away after t days is given by $n(t) = 2t - \frac{1}{8}t^2$, where $0 \le t \le 8$. The cost C in dollars of farming n boxes of mushrooms is given by $C(n) = 20 + 30n$. The wastage due to throw-away boxes of decayed mushrooms can be thought as a composite function.

Inverse of a function may be simply treated as the exchange of roles between the dependent and the independent variables. In terms of notation, the inverse of the function $y = f(x)$ is $x = f^{-1}(y)$ where f^{-1} represents the inverse and not $\frac{1}{f}$. An obvious example is that e^x is the inverse function of ln x, and vice versa. Teachers may explain the condition for the existence of inverse function to abler students.

Teachers should then introduce the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$ and the inverse function rule $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ to find the derivatives of composite functions and inverse functions. Rigorous proofs of these rules are not required but sufficient examples on their applications are expected.

Students should also learn to distinguish explicit functions and implicit functions.

<u>Example 2</u>
The base radius y cm of a solid cylinder having total surface area 24π cm^2 is a function of the height x cm of the cylinder defined implicitly by the relation $y^2+xy-12=0$. By solving this equation, we find that the function y is given explicitly by $y=\frac{-x+\sqrt{x^2+48}}{2}$.

Before introducing implicit differentiation, teachers should give various equations of the form $f(x, y)=0$ that are either difficult or impossible to solve for y explicitly in terms of x, e.g. $x^5-x^3y+y^3-y^4+1=0$.

<u>Example 3</u>
The price \$$P$ and the demand x units for an article are related by the equation $x^2+5xP+25P^2=4500$. The benefit of differentiating this function implicitly can then be observed.
Students are expected to apply the chain rule, inverse function rule and implicit differentiation to solve problems involving simple exponential and logarithmic functions at later stage. However, differentiation of parametric equations is not required.

6.3 Differentiation of e^x and $\ln x$

$\frac{\mathrm{d}}{\mathrm{d}x}e^x=e^x$ may be understood by assuming that it is legitimate to differentiate the infinite series $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$ term by term. By treating $\ln x$ as the inverse function of e^x, $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)=\frac{1}{x}$ may be proved by using the inverse function rule.

Students should make use of logarithmic differentiation to find the derivatives of some complicated functions of x, particularly functions involving variable as indices. Examples may include finding the derivatives of $y=\frac{\sqrt{2x-3}e^{-3x}}{(4x+5)^2}$, $y=e^{x^2}$, $y=\log_a\sqrt{x+1}$ and $y=x^x$.

Students are expected to use the chain rule to deduce the following formulae:

(i) $\frac{\mathrm{d}}{\mathrm{d}x} e^{f(x)} = e^{f(x)} \frac{\mathrm{d}}{\mathrm{d}x} f(x)$;

(ii) $\frac{\mathrm{d}}{\mathrm{d}x} \ln f(x) = \frac{1}{f(x)} \frac{\mathrm{d}}{\mathrm{d}x} f(x)$.

Differentiation of functions like xe^{x^2+1} and $\ln\sqrt{3x^2-5x+7}$ is expected but differentiation of trigonometric functions is not required.

6.4 Second derivative

The notation $f''(x)$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ should be introduced to students. Teachers should emphasise that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \neq \frac{1}{\frac{\mathrm{d}^2 x}{\mathrm{d}y^2}}$. The higher order derivatives are not required.